$$\dot{\mathcal{D}} = \int_{\Omega} \left\{ \dot{\underset{\sim}{F}} \cdot \left(\underset{\sim}{S} - \frac{\partial W}{\partial \underset{\sim}{F}} \right) + \sum{}' \dot{\underset{\sim}{s}}{}^{i} \cdot \left(\mu^{i} - \frac{\partial W}{\partial s^{i}} \right) - \sum \nabla \mu^{i} \cdot j^{i} \right\} dV$$

$$+ \int_{\mathcal{S}} \left\{ \overset{\circ}{\bar{\underset{\sim}{F}}} \cdot \left(\bar{\underset{\sim}{S}} - \frac{\partial W}{\partial \bar{\underset{\sim}{F}}} \right) + \overset{\circ}{\underset{\sim}{m}} \cdot \left(\underset{\sim}{q} - \frac{\partial \psi}{\partial \hat{m}} - \bar{S}^{T} \langle F \rangle \hat{m} \right) \right\} dA$$

$$- \int_{\mathcal{S}} \sum{}' [\![\mu^{i}]\!] \langle \underset{\sim}{j}{}^{i} \rangle \cdot \hat{m} \; dA$$

$$+ \int_{\mathcal{S}} V_{m} \Big(\hat{m} \cdot [\![W \underset{\sim}{I} - \sum{}' s^{i} \langle \mu^{i} \rangle \underset{\sim}{I} - \underset{\sim}{F}^{T} \underset{\sim}{S}]\!] \hat{m}$$
$$+ (\psi \underset{\sim}{P} - \bar{\underset{\sim}{F}}^{T} \bar{S}) \cdot \underset{\sim}{L} + \nabla_{\mathcal{S}} \cdot \underset{\sim}{q} \Big) dA$$

$$+ \int_{\partial \mathcal{S}} \left\{ (q \cdot \hat{w}) V_{m} + \hat{w} \cdot \left(\psi \underset{\sim}{P} - \langle \underset{\sim}{F}^{T} \rangle \bar{S} \right) \hat{w} \; V_{w} \right\} d\ell$$

$$\Longrightarrow \quad \underset{\sim}{S} = \frac{\partial W}{\partial \underset{\sim}{F}}, \quad \mu^{i} = \frac{\partial W}{\partial s^{i}}, \quad j^{i} = -\underset{\sim}{K}^{i} \nabla \mu^{i}$$

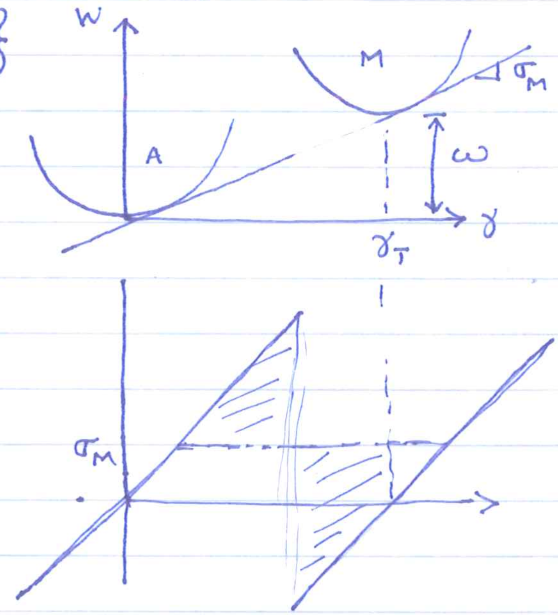
$$\bar{\underset{\sim}{S}} = \frac{\partial W}{\partial \bar{F}}, \quad \underset{\sim}{q} = \frac{\partial \psi}{\partial \hat{m}} - \bar{S}^{T} \langle F \rangle \hat{m}$$

$$\hat{m} \cdot \langle j^{i} \rangle = - \bar{K} [\![\mu^{i}]\!]$$

$$d = \hat{m} \cdot [\![W \underset{\sim}{I} - \sum{}' s^{i} \langle \mu^{i} \rangle \underset{\sim}{I} - \underset{\sim}{F}^{T} \underset{\sim}{S}]\!] + (\psi \underset{\sim}{P} - \bar{\underset{\sim}{F}}^{T} \bar{\underset{\sim}{S}}) \cdot L + \nabla_{\mathcal{S}} \cdot \underset{\sim}{q}$$

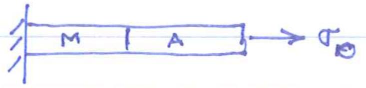
Example 1. Hysteresis in martensite

$$W = \min\left\{ \frac{E}{2}\gamma^2,\ \frac{E}{2}(\gamma-\gamma_T)^2+\omega \right\}$$



$$\sigma = \frac{\partial W}{\partial \gamma} = \begin{cases} E\gamma \\ E(\gamma-\gamma_T) \end{cases}$$

Consider



Equilibrium: $\sigma = \sigma_0$

$$\text{or } \gamma = \begin{cases} E^{-1}\sigma_0 \\ E^{-1}\sigma_0 + \gamma_T \end{cases}$$

Maxwell stress:

$$[\![W - \sigma\gamma]\!] = 0 \text{ or } \frac{E}{2}(E^{-1}\sigma_M)^2 - \frac{E}{2}(E^{-1}\sigma_M + \gamma_T - \gamma_T)^2 - \omega - \sigma_M\left(E^{-1}\sigma_M - E^{-1}\sigma_M - \gamma_T\right) = 0$$

or $\omega - \sigma_M\gamma_T = 0$

or $\boxed{\sigma_M = \frac{\omega}{\gamma_T}}$

Note equal area interpretation

$$[\![W - \sigma\gamma]\!] = \int_{\gamma^-}^{\gamma^+} \{\sigma(\gamma) - \sigma_0\gamma\}\, d\gamma$$

Driving force:

$$d = [\![W - \sigma_0\gamma]\!] = \frac{E}{2}(E^{-1}\sigma_0)^2 - \frac{E}{2}(E^{-1}\sigma_0 + \gamma_T - \gamma_T)^2 - \omega - \sigma_0(E^{-1}\sigma_0 - E^{-1}\sigma_0 - \gamma_T)$$

$$= \sigma_0\gamma_T - \omega$$

$$= \gamma_T(\sigma_0 - \sigma_M)$$

Note 1. Often $\omega(\theta) = \frac{L}{\theta_0}(\theta - \theta_0) \Rightarrow d = 0 \Leftrightarrow \sigma_0 = \frac{L}{\theta_0\gamma_T}(\theta-\theta_0)$ | ... Claussis-Clapeyron

Note 2. $\sigma_0 > \sigma_M \Rightarrow d > 0$ ... prefers martensite

$\sigma_0 < \sigma_m \Rightarrow d < 0$ ... prefer austenite

Nucleation:

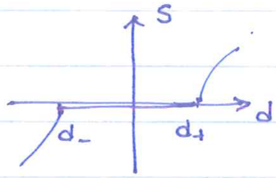
Start with pure phase, put in an infinitesimal interface, calculate driving force and check nucleation criterion:

$d \geq d^+$ put M nucleus in A

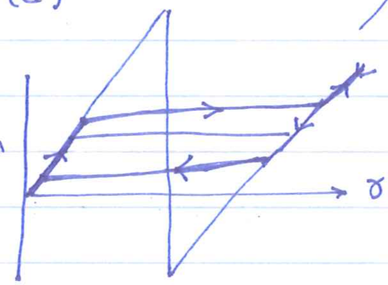
$d \leq d^-$ put A nucleus in M

Kinetic relation:

$\dot{s} = f(d)$



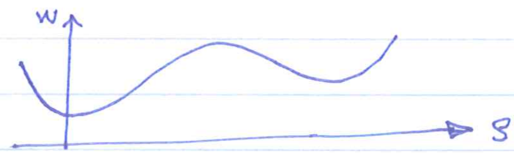
$\Rightarrow \sigma_M$



... Hysteresis loop.

Example 2. Diffusional phase transformation

$W = W(\rho)$



$d = [\![W - \rho\mu]\!] + \sigma\kappa$

$\rho_t = -\nabla\cdot\mu = \nabla\cdot D\nabla\mu$

$[\![\rho]\!] v_m = [\![j]\!]\cdot\hat{m}$

$\hat{m}\cdot\langle j\rangle = K[\![\mu]\!]$ $\qquad v_m = f(d)$

Local equilibrium ... all processes fast compared to v_m:

$d = 0$

$\nabla\cdot D\nabla\mu = 0$

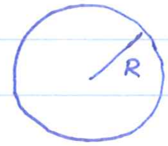
$[\![\mu]\!] = 0$

$[\![\rho]\!] v_m = [\![j]\!]\cdot\hat{m}$

Apply to a single spherical particle

$\nabla\cdot D\nabla\mu = 0$

$\Rightarrow \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\mu = 0$ $\qquad$ Assume spherical symmetry



μ_∞

$\Rightarrow \mu = \frac{A}{r} + B$ for $r > R$, $r < R$

$\Rightarrow \mu = \begin{cases} \bar{\mu} & \text{for } r < R \\ \mu_\infty - (\mu_\infty - \mu_R)\frac{R}{r} & \text{for } r \geq R \end{cases}$ $\qquad$ Also, $\bar{\mu} = \mu_R$